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The
Linear model. I

The "fixed" effects model

- Gibbs sampling
  - Flat priors
  - Inverted gamma priors
- Least squares as a Bayesian method

The "fixed" effects model

NOTATION

A matrix

b vector

c scalar

f density

The "fixed" effects model

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$$

$$\mathbf{y} \sim N(\mathbf{X}\mathbf{b}, \mathbf{I}\sigma^2)$$

$$f(\mathbf{b}|\mathbf{y}) = \int f(\mathbf{b},\sigma^2|\mathbf{y})\, d\sigma^2$$

$$f(\sigma^2|\mathbf{y}) = \int f(\mathbf{b},\sigma^2|\mathbf{y})\, d\mathbf{b}$$

Gibbs sampling

OBJECTIVE: samples of $f(\sigma^2|\mathbf{y})$ and $f(\mathbf{b}|\mathbf{y})$

σ_0^2 arbitrary

$f(\mathbf{b} \mid \sigma^2 = \sigma_0^2, \mathbf{y})$

$f(\sigma^2 \mid \mathbf{b} = \mathbf{b}_0, \mathbf{y})$

Normal distribution

$$f\left(\mathbf{y}\,|\,\mathbf{b},\sigma^2\right) \propto \frac{1}{\left|\mathbf{I}\sigma^2\right|^{\frac{1}{2}}} \exp\left[-\frac{\left(\mathbf{y}-\mathbf{Xb}\right)'\left[\mathbf{I}\sigma^2\right]^{-1}\left(\mathbf{y}-\mathbf{Xb}\right)}{2}\right]$$

$$\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{y}-\mathbf{Xb}\right)'\left(\mathbf{y}-\mathbf{Xb}\right)}{2\sigma^2}\right]$$

Conditional distributions

$$f\left(\mathbf{b} \mid \mathbf{y}, \sigma^2\right) \propto f\left(\mathbf{y} \mid \mathbf{b}, \sigma^2\right) f\left(\mathbf{b}\right) \propto f\left(\mathbf{y} \mid \mathbf{b}, \sigma^2\right)$$

f(**b**)=constant

$$\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)'\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)}{2\sigma^2}\right]$$

Conditional distributions

$$f\left(\sigma^2 \,|\, \mathbf{y},\mathbf{b}\right) \propto f\left(\mathbf{y} \,|\, \sigma^2,\mathbf{b}\right) f\left(\sigma^2\right) \propto f\left(\mathbf{y} \,|\, \sigma^2,\mathbf{b}\right)$$

f(σ^2)=constant

$$\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)'\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)}{2\sigma^2}\right]$$

Conditional distributions

$$f\left(\mathbf{b} \mid \mathbf{y}, \sigma^2\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{(\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})}{2\sigma^2}\right]$$

NOT NORMAL!!

$$f\left(\sigma^2 \mid \mathbf{b}, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{(\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})}{2\sigma^2}\right]$$

Conditional distributions

$$f\left(\sigma^2 \mid \mu, \mathbf{y}\right) = \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\sum_1^n \left(y_i - \mu\right)^2}{2\sigma^2}\right]$$

INVERTED GAMMA $\beta = \frac{1}{2}\sum_1^n \left(y_i - \mu\right)^2$ $\alpha = \frac{n}{2} - 1$

$$f\left(\sigma^2 \mid \mathbf{b}, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{y} - \mathbf{Xb}\right)'\left(\mathbf{y} - \mathbf{Xb}\right)}{2\sigma^2}\right]$$

Conditional distributions

$$f\left(\sigma^2 \,|\, \mathbf{b}, \mathbf{y}\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{y}-\mathbf{Xb}\right)'\left(\mathbf{y}-\mathbf{Xb}\right)}{2\sigma^2}\right]$$

INVERTED GAMMA $\beta = \frac{1}{2}\left(\mathbf{y}-\mathbf{Xb}\right)'\left(\mathbf{y}-\mathbf{Xb}\right)$ $\alpha = \frac{n}{2} - 1$

Conditional distributions

$$f\left(\mathbf{b} \mid \mathbf{y}, \sigma^2\right) \propto f\left(\mathbf{y} \mid \mathbf{b}, \sigma^2\right) f\left(\mathbf{b}\right) \propto f\left(\mathbf{y} \mid \mathbf{b}, \sigma^2\right)$$

f(b)=constant

$$\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)'\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)}{2\sigma^2}\right]$$

Conditional distributions

$$(\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b}) = \mathbf{y}'\mathbf{y} - 2\mathbf{b}'\mathbf{X}'\mathbf{y} + \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}$$

$$\mathbf{b}'\mathbf{X}'\mathbf{y} = \mathbf{y}'\mathbf{X}\mathbf{b}$$

$$\hat{\mathbf{b}} = [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y}$$

$$\mathbf{X}'\mathbf{X}\hat{\mathbf{b}} = \mathbf{X}'\mathbf{y}$$

$$(\mathbf{b}-\hat{\mathbf{b}})'\mathbf{X}'\mathbf{X}(\mathbf{b}-\hat{\mathbf{b}}) = \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} - 2\mathbf{b}'\mathbf{X}'\mathbf{X}\hat{\mathbf{b}} + \hat{\mathbf{b}}'\mathbf{X}'\mathbf{X}\hat{\mathbf{b}}$$

$$= \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} - 2\mathbf{b}'\mathbf{X}'\mathbf{y} + \mathbf{y}'\mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y}$$

Conditional distributions

$$(\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b}) = \mathbf{y}'\mathbf{y} - 2\mathbf{b}'\mathbf{X}'\mathbf{y} + \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}$$

$$\left(\mathbf{b}-\hat{\mathbf{b}}\right)'\mathbf{X}'\mathbf{X}\left(\mathbf{b}-\hat{\mathbf{b}}\right) = \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} - 2\mathbf{b}'\mathbf{X}'\mathbf{y} + \mathbf{y}'\mathbf{X}\left[\mathbf{X}'\mathbf{X}\right]^{-1}\mathbf{X}'\mathbf{y}$$

$$\begin{aligned}(\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b}) &= \\ &= \left(\mathbf{b}-\hat{\mathbf{b}}\right)'\mathbf{X}'\mathbf{X}\left(\mathbf{b}-\hat{\mathbf{b}}\right) + \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}\left[\mathbf{X}'\mathbf{X}\right]^{-1}\mathbf{X}'\mathbf{y} \\ &= \left(\mathbf{b}-\hat{\mathbf{b}}\right)'\mathbf{X}'\mathbf{X}\left(\mathbf{b}-\hat{\mathbf{b}}\right) + \text{constant}\end{aligned}$$

Conditional distributions

$$f\left(\mathbf{b} \mid \mathbf{y}, \sigma^2\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)'\left(\mathbf{y}-\mathbf{X}\mathbf{b}\right)}{2\sigma^2}\right]$$

$$\propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{b}-\hat{\mathbf{b}}\right)'\mathbf{X}'\mathbf{X}\left(\mathbf{b}-\hat{\mathbf{b}}\right)}{2\sigma^2}\right]$$

Conditional distributions

$$f(\mathbf{y}\,|\,\mathbf{m},\mathbf{V}) = \frac{1}{|V|^{1/2}} \exp\left[-\frac{(\mathbf{y}-\mathbf{m})'\,\mathbf{V}^{-1}(\mathbf{y}-\mathbf{m})}{2}\right] \qquad \sim N(\mathbf{m},\mathbf{V})$$

$$f\left(\mathbf{b}\,|\,\mathbf{y},\sigma^2\right) \propto \frac{1}{\left(\sigma^2\right)^{\frac{n}{2}}} \exp\left[-\frac{\left(\mathbf{b}-\hat{\mathbf{b}}\right)'\,\mathbf{X}'\mathbf{X}\left(\mathbf{b}-\hat{\mathbf{b}}\right)}{2\sigma^2}\right] \qquad \sim N\left(\hat{\mathbf{b}},\,(\mathbf{X}'\mathbf{X})^{-1}\sigma^2\right)$$

LS is a Bayesian method

$$f\left(\mathbf{b} \mid \sigma^2, \mathbf{y}\right) \sim N\left(\hat{\mathbf{b}}, \left(\mathbf{X}'\mathbf{X}\right)^{-1} \sigma^2\right)$$

$$\begin{aligned} LS\left(\mathbf{b}\right) = \hat{\mathbf{b}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{y} &= \mathit{mode}\, f\left(\mathbf{b} \mid \sigma^2, \mathbf{y}\right) \\ &= \mathit{median}\, f\left(\mathbf{b} \mid \sigma^2, \mathbf{y}\right) \\ &= \mathit{mean}\, f\left(\mathbf{b} \mid \sigma^2, \mathbf{y}\right) \end{aligned}$$

…it does not take into account the error of estimation of σ^2

Gibbs sampling

OBJECTIVE: samples of $f(\sigma^2|\mathbf{y})$ and $f(\mathbf{b}|\mathbf{y})$

σ_0^2 arbitrary

$$f(\mathbf{b} \mid \sigma^2 = \sigma_0^2, \mathbf{y})$$

$$f(\sigma^2 \mid \mathbf{b} = \mathbf{b}_0, \mathbf{y})$$